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# Characterization of the Strong Closure of $C^\infty(\boldsymbol{B}^4; \mathbf{S}^2)$ in $W^{1,p}(\boldsymbol{B}^4; \mathbf{S}^2)$ $(\frac{16}{5} \le p < 4)$

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## 1. INTRODUCTION AND STATEMENT OF THE RESULT

Let $M$ and $N$ be two compact Riemannian manifolds. We may assume, by the Nash theorem, that $N$ is a submanifold of the Euclidean space $\mathbf{R}^k$ for some $k > 1$.

We denote by $C^\infty(M; N)$ the space of smooth mappings between $M$ and $N$ and by $W^{1,p}(M; N)$ the space of Sobolev mappings, that is,

$$W^{1,p}(M; N) := \{u \in W^{1,p}(M; \mathbf{R}^k) : u(x) \in N \ a.e.\},$$

where $W^{1,p}(M; \mathbf{R}^k)$ is the standard Sobolev space.

It is known that $C^\infty(M; N)$ is not necessarily strongly dense in $W^{1,p}(M; N)$ for some $p$ $(1 \le p < \infty)$. More precisely, we have the following:

THEOREM IN [2, 3, AND 8]. *$C^\infty(M; N)$ is strongly dense in $W^{1,p}(M; N)$ if and only if* (i) $p \ge \dim M$ *or* (ii) $1 \le p < \dim M$ *and* $\pi_{[p]}(N) = 0$. *Here $\pi_{[p]}(N)$ stands for a $[p]$th homotopy group of $N$ and $[p]$ is the largest integer less than or equal to $p$.*

Therefore, it is of interest to characterize $\hat{W}^{1,p}(M; N)$, the strong closure of $C^\infty(M; N)$ in $W^{1,p}(M; N)$, when $1 \le p < \dim M$ and $\pi_{[p]}(N) \ne 0$. This is called the density problem of $C^\infty(M; N)$ in $W^{1,p}(M; N)$.

In the following, $\boldsymbol{B}^m$(or $\overline{\boldsymbol{B}}^m$) denotes the unit open (or closed) ball in $\mathbf{R}^m$ and $\mathbf{S}^m = \overline{\boldsymbol{B}}^m \backslash \boldsymbol{B}^m$.



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Concerning the above problem, some partial results have been obtained. The first result in that direction was given by Bethuel [1], who characterizes $\hat{W}^{1,p}(\boldsymbol{B}^3; \mathbf{S}^2)$ for $2 \leq p < 3$, that is,

$$\hat{W}^{1,p}(\boldsymbol{B}^3; \mathbf{S}^2) = \{u \in W^{1,p}(\boldsymbol{B}^3; \mathbf{S}^2) : \operatorname{div} D(u) = 0\},$$

where $D(u) = (u \cdot u_{x_2} \wedge u_{x_3},\ u \cdot u_{x_3} \wedge u_{x_1},\ u \cdot u_{x_1} \wedge u_{x_2})$.

Note that $\hat{W}^{1,p}(\boldsymbol{B}^3; \mathbf{S}^2) = W^{1,p}(\boldsymbol{B}^3; \mathbf{S}^2)$ when $1 \leq p < 2$ or $p \geq 3$ by the result of [2] and [8].

On the other hand, Demengel [6] characterizes $\hat{W}^{1,p}(\boldsymbol{B}^m; \mathbf{S}^1)$ as follows:

$$\hat{W}^{1,p}(\boldsymbol{B}^m; \mathbf{S}^1) = \{u \in W^{1,p}(\boldsymbol{B}^m; \mathbf{S}^1) : \operatorname{curl} H(u) = 0\},$$

where $H(u) = (u \wedge u_{x_1},\ u \wedge u_{x_2},\ \ldots,\ u \wedge u_{x_m})$.

We should remark also that $\hat{W}^{1,p}(\boldsymbol{B}^m; \mathbf{S}^1) = \hat{W}^{1,p}(\boldsymbol{B}^m; \mathbf{S}^1)$ when $p \geq 2$.

Some parts of the above results were unified by Bethuel *et al.* [4].

THEOREM IN [4]. *Assume that $N$ is $([p]-1)$-connected (i.e., $\pi_i(N) = 0$ for $0 \leq i \leq [p]-1$) and $H_{[p]}(N; \mathbf{Z})$ is torsion free (if $[p] = 1$, then we assume that only $\pi_1(N)$ is Abelian). Then, $u \in \hat{W}^{1,p}(M; N)$ if and only if $u^*\omega$ is closed for any $\omega$ in $\wedge^{[p]} T^*N$ with $d\omega = 0$. Here $H_{[p]}(N; \mathbf{Z})$ is a $[p]$th singular homology group of $N$ with coefficient in $\mathbf{Z}$ and $u^*\omega$ represents the pull back $[p]$-form of $\omega$ by $u$.*

This theorem excludes the following fact given by [4, Remark f]: There exists a map $u$ in $W^{1,p}(\boldsymbol{B}^4; \mathbf{S}^2)$ $(3 \leq p < 4)$ which cannot be approximated by smooth maps, nevertheless it satisfies $d(u^*\omega) = 0$ for any $\omega$ in $\wedge^k T^*\mathbf{S}^2$ $(k = 1$ or $k = 2)$ with $d\omega = 0$.

This fact does not contradict the above theorem because $\pi_2(\mathbf{S}^2) \neq 0$.

Moreover, in Section 2, we show that for any $u \in W^{1,p}(\boldsymbol{B}^4; \mathbf{S}^2)$ $(3 \leq p < 4)$, we have $d(u^*\omega) = 0$ for any $\omega$ in $\wedge^k T^*\mathbf{S}^2$ $(k = 1$ or $k = 2)$ with $d\omega = 0$.

Summing up, we have

$$\hat{W}^{1,p}(\boldsymbol{B}^4; \mathbf{S}^2) = \hat{W}^{1,p}(\boldsymbol{B}^4; \mathbf{S}^2) \qquad \text{for } 1 \leq p < 2 \text{ or } p \geq 4,$$

$$\hat{W}^{1,p}(\boldsymbol{B}^4; \mathbf{S}^2) = \{u \in \hat{W}^{1,p}(\boldsymbol{B}^4; \mathbf{S}^2) : d(u^*\omega) = 0$$

$$\text{for any } \omega \in \wedge^2 T^*\mathbf{S}^2 \text{ with } d\omega = 0\} \qquad \text{for } 2 \leq p < 3.$$

Our aim is to characterize $\hat{W}^{1,p}(\boldsymbol{B}^4; \mathbf{S}^2)$ for $3 \leq p < 4$ but unfortunately only for $\frac{16}{5} \leq p < 4$ for the time being. In order to state our result, we prepare some notation.

For a given smooth map $\varphi: \overline{\boldsymbol{B}}^4 \to \mathbf{S}^2$, we set

$$C^\infty_\varphi(\overline{\boldsymbol{B}}^4; \mathbf{S}^2) = \{u \in C^\infty(\overline{\boldsymbol{B}}^4; \mathbf{S}^2) : u|_{\partial \boldsymbol{B}^4} = \varphi|_{\partial \boldsymbol{B}^4}\}$$

and

$$W^{1,p}_{\varphi}(\boldsymbol{B}^4; \mathbf{S}^2) = \{u \in W^{1,p}(\boldsymbol{B}^4; \mathbf{S}^2) \colon u|_{\partial \boldsymbol{B}^4} = \varphi\,|_{\partial \boldsymbol{B}^4}\}.$$

Our main result is the following:

THEOREM. *Let $p$ be a real number satisfying $\frac{16}{5} \le p < 4$. Let $\varphi\colon \boldsymbol{B}^4 \to \mathbf{S}^2$ be a given smooth map. Then $u \in W^{1,p}_{\varphi}(\boldsymbol{B}^4; \mathbf{S}^2)$ can be approximated by maps in $C^\infty_{\varphi}(\boldsymbol{B}^4; \mathbf{S}^2)$ if and only if*

$$(u^*\omega) \wedge (u^*\omega) = 0 \qquad \text{in } \mathcal{D}'(\wedge^4 T^* \boldsymbol{B}^4),$$

*where $\omega$ is a generator of the second de Rham cohomology group of $\mathbf{S}^2$ and $\mathcal{D}'(\wedge^4 T^*\boldsymbol{B}^4)$ is the space of 4-form valued distributions on $\boldsymbol{B}^4$.*

*Remarks*. (a) It is not obvious to give the meaning of $(u^*\omega) \wedge (u^*\omega)$ for every $u$ in $W^{1,p}(\boldsymbol{B}^4; \mathbf{S}^2)$. But we show in Section 2 that this quantity is meaningful as a current when $\frac{16}{5} \le p < 4$.

(b) Some extensions to the case where the base manifold $\boldsymbol{B}^4$ is replaced by a general four-dimensional Riemannian manifold $M$ with or without boundary are remarked at the end of Section 2.

(c) We have not yet obtained definitive result in the case $3 \le p < \frac{16}{5}$.

(d) It seems worthwhile to note that the characterizing condition of $W^{1,p}(\boldsymbol{B}^4; \mathbf{S}^2)$ has a remarkable difference between $2 \le p < 3$ and $\frac{16}{5} \le p < 4$.

## 2. PROOF OF THEOREM

In this section, we give the proof of our main theorem as stated in Section 1.

We first give the precise meaning of $(u^*\omega) \wedge (u^*\omega)$ for $u$ in $W^{1,p}_{\varphi}(\boldsymbol{B}^4; \mathbf{S}^2)$ as a distribution. For this purpose, we need some lemmas.

The proof of the following lemma is easy, but the result is essential in this paper.

LEMMA 2.1 *For any $u \in W^{1,p}(\boldsymbol{B}^4; \mathbf{S}^2)$ $(3 \le p < 4)$ and $\omega \in \wedge^k T^*\mathbf{S}^2$ $(k = 1$ or $k = 2)$ with $d\omega = 0$, we have*

$$d(u^*\omega) = 0 \qquad \text{in } \mathcal{D}'(\wedge^{k+1} T^*\boldsymbol{B}^4).$$

*Proof.* We prove $u$ can be strongly approximated by smooth maps in $W^{1,2}$. In fact, by the density theorem of Bethuel [2, Theorem 2], $u$ can be approximated in $W^{1,p}$ by maps which are smooth except for finitely many points in $\boldsymbol{B}^4$. So, in particular, $u$ can be approximated in $W^{1,2}$ by maps which are smooth except at most finitely many points in $\boldsymbol{B}^4$. On the other hand, by the result of [2, Lemma 2 (i)], the map in $W^{1,2}$ which has only finitely many singularities can be approximated by smooth ones in $W^{1,2}$. So $u$ can be approximated in $W^{1,2}$ by smooth maps.

Let $u_n \in C^\infty(\boldsymbol{B}^4; \mathbf{S}^2)$ be such that $u_n \to u$ in $W^{1,2}$. Then we have

$$u_n^*\omega \to u^*\omega \qquad \text{in } L^1(\wedge^k T^*\boldsymbol{B}^4).$$

Thus

$$d(u_n^*\omega) \to d(u^*\omega) \qquad \text{in } \mathcal{D}'(\wedge^{k+1}T^*\boldsymbol{B}^4).$$

Since $u_n$ is smooth, we have $d(u_n^*\omega) = 0$ and we finally obtain $d(u^*\omega) = 0$. This completes the proof of Lemma 2.1. ∎

We also need the following lemma which essentially follows from the $L^p$-version of the de Rham–Hodge–Kodaira decomposition theorem.

Here and in the following, $\omega$ denotes a generator of the second de Rham cohomology group $H^2_{\mathrm{DR}}(\mathbf{S}^2)$ of $\mathbf{S}^2$.

For convenience in the proof, we assume any $u \in W^{1,p}_\varphi(\boldsymbol{B}^4; \mathbf{S}^2)$ is defined in $\boldsymbol{B}^4_{1+\rho} = \{x \in \mathbf{R}^4 : |x| < 1+\rho\}$ for some $\rho > 0$ by $u(x) = \varphi(x/|x|)$ in $\boldsymbol{B}^4_{1+\rho} \backslash \boldsymbol{B}^4$.

LEMMA 2.2. *Let $u$ be an arbitrary element in $W^{1,p}_\varphi(\boldsymbol{B}^4; \mathbf{S}^2)$. Then there exists a 1-form $\omega(u)$ in $W^{1,p}(\wedge^1 T^*\boldsymbol{B}^4_{1+\rho})$ such that $u^*\omega = d\omega(u) + \varphi^*\omega$ and $d^*\omega(u) = 0$, where $d^*$ is the formal adjoint of $d$ in the standard metric on $\boldsymbol{B}^4$.*

*Moreover, for any $u, v \in W^{1,p}_\varphi(\boldsymbol{B}^4; \mathbf{S}^4)$, there exists a closed 1-form $\omega(u, v) \in W^{1,p/2}(\wedge^1 T^*\boldsymbol{B}^4_{1+\rho})$ such that the following holds:*

$$\|\omega(u) - \omega(v) + \omega(u, v)\|_{W^{1,p/2}(\boldsymbol{B}^4_{1+\rho})} \le C\|u^*\omega - v^*\omega\|_{L^{p/2}(\boldsymbol{B}^4)}.$$

*Proof.* By Lemma 2.1, we have the following:

$$\begin{cases} d(u^*\omega - \varphi^*\omega) = 0 & \text{in } \boldsymbol{B}^4_{1+\rho} \\ u^*\omega - \varphi^*\omega|_{\partial \boldsymbol{B}^4_{1+\rho}} = 0. \end{cases}$$

So by the $L^p$-version of the de Rham–Hodge–Kodaira decomposition (see [7] and the Appendix, Corollary A.2), there exists some $\omega(u) \in W^{1,p/2}(\wedge^1 T^*\boldsymbol{B}^4_{1+\rho})$ such that $d\omega(u) = u^*\omega - \varphi^*\omega$. Thus we obtain

$$d(\omega(u) - \omega(v)) = u^*\omega - v^*\omega. \tag{1}$$

Since $u^*\omega - v^*\omega|_{\partial \boldsymbol{B}^4_{1+\rho}} = 0$ and $d(u^*\omega - v^*\omega) = 0$, by Corollary A.2 on the Appendix there exists some $\alpha(u, v) \in W^{1,p/2}(\wedge^1 T^*\boldsymbol{B}^4_{1+\rho})$ satisfying

$$d\alpha(u, v) = u^*\omega - v^*\omega \tag{2}$$

and

$$\|\alpha(u, v)\|_{W^{1,p/2}(\boldsymbol{B}^4_{1+\rho})} \le C\|u^*\omega - v^*\omega\|_{L^{p/2}(\boldsymbol{B}^4)}. \tag{3}$$

Put $\omega(u, v) = \alpha(u, v) - (\omega(u) - \omega(v))$. Then by (1) and (2) we have $d\omega(u, v) = 0$, $\omega(u, v) \in W^{1,p/2}(\wedge^1 T^*\boldsymbol{B}^4_{1+\rho})$, and

$$\|\omega(u) - \omega(v) + \omega(u, v)\|_{W^{1,p/2}(\boldsymbol{B}^4_{1+\rho})} \le C\|u^*\omega - v^*\omega\|_{L^{p/2}(\boldsymbol{B}^4)}.$$

This completes the proof. ∎

We are now ready to give a meaning of $(u^*\omega) \wedge (u^*\omega)$ as a distribution. By Lemma 2.1 and Lemma 2.2, we can formally write

$$(u^*\omega) \wedge (u^*\omega) = d\{\omega(u) \wedge (u^*\omega)\} + (\varphi^*\omega) \wedge (u^*\omega). \tag{\#}$$

Since $(\varphi^*\omega) \wedge (u^*\omega) \in W^{1,p/2}(\wedge^4 T^*\boldsymbol{B}^4) \subset \mathcal{D}'(\wedge^4 T^*\boldsymbol{B}^4)$, if we can give a meaning $d\{\omega(u) \wedge (u^*\omega)\}$ as a distribution, then we may define $(u^*\omega) \wedge (u^*\omega)$ by the formula (#).

By the Sobolev embedding theorem, we have $\omega(u) \in W^{1,p/2} \subset L^{4p/(8-p)}$. If $\frac{16}{5} \le p < 4$, then $L^{4p/(8-p)} \subset L^{p/(p-2)} = (L^{p/2})^*$, where $(L^{p/2})^*$ is the dual of $L^{p/2}$. Since $u^*\omega \in L^{p/2}$, we obtain $\omega(u) \wedge (u^*\omega) \in L^1$ and we can give a meaning $d\{\omega(u) \wedge (u^*\omega)\}$ as a distribution.

We return to the proof of our main theorem. We first prove necessity.

PROPOSITION 2.3. *If $u \in W^{1,p}(\boldsymbol{B}^4; \mathbf{S}^2)$ $(\frac{16}{5} \le p < 4)$ can be strongly approximated by maps in $C^\infty_\varphi(\boldsymbol{B}^4; \mathbf{S}^2)$, then we have*

$$(u^*\omega) \wedge (u^*\omega) = 0 \qquad \text{in } \mathcal{D}'(\wedge^4 T^*\boldsymbol{B}^4).$$

*Proof.* Let $u_n \in C^\infty_\varphi(\boldsymbol{B}^4; \mathbf{S}^2)$ be such that $u_n \to u$ in $W^{1,p}$. By Lemma 2.2, there exists a sequence of closed 1-forms $\{\omega_n\}$ $(\omega_n = \omega(u_n, u)) \subset W^{1,p/2}(\wedge^1 T^*\boldsymbol{B}^4_{1+\rho})$ such that

$$\omega(u_n) - \omega(u) + \omega_n \to 0 \qquad \text{in } W^{1,p/2}.$$

By the Sobolev embedding theorem, if $\frac{16}{5} \le p < 4$, we have

$$\omega(u_n) - \omega(u) + \omega_n \to 0 \qquad \text{in } L^{p/(p-2)} = (L^{p/2})^*. \tag{4}$$

Thus we obtain

$$\begin{aligned}0 &= (u_n^*\omega) \wedge (u_n^*\omega) = (d\omega(u_n) + \varphi^*\omega) \wedge (u_n^*\omega)\\ &= d\omega(u_n) \wedge (u_n^*\omega) + (\varphi^*\omega) \wedge (u_n^*\omega)\\ &= d(\omega(u_n) + \omega_n) \wedge (u_n^*\omega) + (\varphi^*\omega) \wedge (u_n^*\omega)\\ &= d\{(\omega(u_n) + \omega_n) \wedge (u_n^*\omega)\} + (\varphi^*\omega) \wedge (u_n^*\omega)\\ &\quad \to d\{\omega(u) \wedge (u^*\omega)\} + (\varphi^*\omega) \wedge (u^*\omega)\\ &= (u^*\omega) \wedge (u^*\omega) \text{ in } \mathcal{D}'(\wedge^4 T^*\boldsymbol{B}^4).\end{aligned}$$

Here we remark that for $u \in C^\infty(\boldsymbol{B}^4; \mathbf{S}^2)$, we have

$$(u^*\omega) \wedge (u^*\omega) = u^*(\omega \wedge \omega) = 0,$$

since $\omega \wedge \omega$ is a 4-form on $\mathbf{S}^2$ and is 0. This completes the proof. ∎

Next we prove sufficiency. For that purpose, we first recall a relation between $\pi_3(\mathbf{S}^2)$ and the Hopf-invariant.

DEFINITION 2.4. Let $u : \mathbf{S}^3 \to \mathbf{S}^2$ be a smooth map and let $\omega$ be a generator of $H^2_{\mathrm{DR}}(\mathbf{S}^2)$ such that $\int_{\mathbf{S}^2} \omega = 1$.

Since $H^2_{\mathrm{DR}}(\mathbf{S}^2) = 0$, there exists a 1-form $\beta$ on $\mathbf{S}^3$ such that $u^*\omega = d\beta$. We define the *Hopf-invariant* $H(u)$ of $u$ by the following expression:

$$H(u) = \int_{\mathbf{S}^3} \beta \wedge d\beta.$$

It is known that $H(u)$ is an integer and $H(u)$ is independent of the choice of $\beta$. Moreover, $H(u)$ only depends on the homotopy class of $u$ and $H : \pi_3(\mathbf{S}^2) \to \mathbf{Z}$ is an isomorphism. For more details, see [5].

We return to the proof of sufficiency.

PROPOSITION 2.5. *Let* $u \in W^{1,p}_\varphi(\boldsymbol{B}^4; \mathbf{S}^2)$ $(\frac{16}{5} \le p < 4)$ *satisfy* $(u^*\omega) \wedge (u^*\omega) = 0$ *in* $\mathcal{D}'(\wedge^4 T^*\boldsymbol{B}^4)$. *Then $u$ can be strongly approximated by smooth maps in* $W^{1,p}$.

*Proof.* We follow the construction of Bethuel [2]. We may assume without loss of generality that the domain is a unit cube $C := [0, 1]^4$ in $\mathbf{R}^4$.

Let us choose a real number $a_i$ in $[1/4n, 3/4n]$ for $n \in \mathbf{N}$ and $i \in \{1, .., 4\}$ such that

$$2n \int_C |\nabla u|^p \, dx \ge \sum_{k=0}^{n-1} \int_{P(a_i + k/n,\, e_i)} |\nabla u|^p \, dx$$

and

$$u\,|_{P(a_i+k/n,\, e_i)} \in W^{1,p}\left(P\left(a_i + \frac{k}{n}, e_i\right)\right) \subset C^0,$$

where $P(a_i + k/n, e_i)$ = three-dimensional plane orthogonal to $e_i$ (=$i$th vector of the canonical Euclidean basis of $\mathbf{R}^4$) which contains the point $(a_i + k/n)\, e_i$.

The existence of such points is proved by using Fubini's theorem (see [2] for details).

Slicing the cube $C$ by planes $P(a_i + k/n, e_i)$ ($i = 1, \ldots, 4$, $k = 0, \ldots, n - 1$), we see that $C$ is divided into $(n + 1)^4$ cubes $C_r$ ($r = 1, \ldots, (n + 1)^4$). Next we decompose $C_r$ into two categories, "good cubes" and "bad cubes."

A *good cube* is a cube $C_r$ satisfying

$$n^{4-p} \int_{C_r} |\nabla u|^p \, dx \leq \varepsilon\, n^{-v} \qquad \text{and} \qquad n^{3-p} \int_{\partial C_r} |\nabla u|^p \, dx \leq \varepsilon,$$

where $\varepsilon > 0$ and $v > 0$ are fixed small constants as in [2, p 159].

A *bad cube* is a cube which is not a good cube.

By the construction of Bethuel [2], there exists a sequence $\{u_k\}$ defined in $Q$, where $Q$ is a union of good cubes, such that $u_k|_{\partial C_r} = u|_{\partial C_r}$ for $C_r$: good cubes, $u_k$ is continuous in $Q$, and

$$\int_Q |\nabla u - \nabla u_k|^p dx \to 0 \qquad \text{as } k \to \infty.$$

On a bad cube $C_r$, by [2] there exists a sequence $\{\bar{u}_k\}$ defined in $P$, where $P$ is a union of bad cubes, such that $\bar{u}_k$ is continuous except for a finite set $\{x_p\}$, where $x_p$ is the barycenter of subcube $C_{r,p} \subset C_r$ satisfying

$$\bigcup_{p=1}^{(q+1)^4} C_{r,p} = C_r, \qquad u|_{\partial C_{r,p}} \in W^{1,p} \subset C^0,$$

$$\bar{u}_k | C_{r,p} = \text{radial extension of } u|_{\partial C_{r,p} \text{ to } C_{r,p}},$$

and

$$\int_P |\nabla u - \nabla \bar{u}_k|^p dx \to 0 \qquad \text{as } k \to \infty.$$

Thus it is sufficient to prove that $\bar{u}_k$ can be approximated by smooth maps.

By Bethuel and Zheng [3, Theorem 5] and its proof, it is sufficient to prove that $u|_{\partial C_{r,p}}: \partial C_{r,p} \to S^2$ is 0-homotopic. For convenience we assume $C_{r,p} = [-1/h, 1/h]^4$ for some $h > 0$. Since $H^2_{\mathrm{DR}}(\boldsymbol{B}^4) = 0$, there exists some $\omega_\varphi \in \wedge^1 T^* \boldsymbol{B}^4$ such that $d\omega_\varphi = \varphi^*\omega$. Since

$$\begin{aligned} 0 &= (u^*\omega) \wedge (u^*\omega) \\ &= (d\omega(u) + \varphi^*\omega) \wedge (u^*\omega) \\ &= d((\omega(u) + \omega_\varphi) \wedge (u^*\omega)) \qquad \text{(by Lemma 2.1)}, \end{aligned}$$

we have

$$\int_{B^4} ((\omega(u) + \omega_\varphi) \wedge u^*\omega)) \wedge d\alpha = 0 \tag{5}$$

for all $\alpha \in C_0^\infty(\boldsymbol{B}^4)$. We take

$$\alpha(x) = \begin{cases} 1 & \text{if } \|x\| \le \dfrac{1-\lambda}{h} \\ \dfrac{1 - h\|x\|}{\lambda} & \text{if } \dfrac{1-\lambda}{h} \le \|x\| \le \dfrac{1}{h}, \end{cases}$$

where $\|x\| := \max_{1\le i\le 4} |x_i|$.

Approximating $\alpha$ by the elements of $C_0^\infty(\boldsymbol{B}^4)$, we have

$$\frac{h}{\lambda} \int_{(1-\lambda)/h \le \|x\| \le 1/h} ((\omega(u) + \omega_\varphi) \wedge (u^*\omega)) \wedge d\|x\| = 0.$$

Letting $\lambda \to 0$, we obtain (we can always assume that for a cube $C_{r,p}$ the limit exists)

$$\int_{\partial C_{r,p}} (i^*\omega(u) + i^*\omega_\varphi) \wedge i^*(u^*\omega) = 0, \tag{6}$$

where $i: \partial C_{r,p} \to \mathbf{R}^4$ is an inclusion.

On the other hand, since $u|_{\partial C_{r,p}} \in W^{1,p} \cap C^0$, $u$ can be approximated by smooth maps in $W^{1,p}(\partial C_{r,p})$ and in $C^0(\partial C_{r,p})$ (this follows from the standard approximation method using a mollifier). Let $\{u_i\} \subset W^{1,p} \cap C^0(\partial C_{r,p}; \mathbf{S}^2)$ be such an approximating sequence of $u$. Since $d(u_i^*\omega - i^*\varphi^*\omega) = 0$ on $\partial C_{r,p}$, by Corollary A.3 there exist 1-forms $\omega_i \in C^\infty(\wedge^1 T^* \partial C_{r,p})$ satisfying

$$d\omega_i = u_i^*\omega - i^*\varphi^*\omega \qquad \text{on } \partial C_{r,p} \text{ and } d^*\omega_i = 0.$$

Since

$$d(i^*\omega(u)) = i^*d\omega(u) = i^*u^*\omega - i^*\varphi^*\omega,$$

we have

$$d(\omega_i - i^*\omega(u)) = u_i^*\omega - i^*u^*\omega \qquad \text{on } \partial C_{r,p}.$$

Therefore by Corollary A.3, there exist closed forms $\alpha_i$ such that

$$\|\omega_i - i^*\omega(u) + \alpha_i\|_{W^{1,p/2}(\partial C_{r,p})} \le C\|u_i^*\omega - i^*u^*\omega\|_{L^{p/2}(\partial C_{r,p})} \to 0 \qquad \text{as } i \to \infty.$$

Thus by (6), we obtain

$$\int_{\partial C_{r,p}} (\omega_i + i^*\omega_\varphi + \alpha_i) \wedge u_i^*\,\omega = \int_{\partial C_{r,p}} (\omega_i + i^*\omega_\varphi) \wedge u_i^*\,\omega$$
$$\to \int_{\partial C_{r,p}} (i^*\omega(u) + i^*\,\omega_\varphi) \wedge (i^*u^*\omega) = 0. \tag{7}$$

On the other hand, we have

$$d(\omega_i + i^*\omega_\varphi) = d\omega_i + i^*d\omega_\varphi = u_i^*\omega.$$

So the left-hand side of (7) is the Hopf invariant of $u_i$: $\partial C_{r,p} \cong \mathbf{S}^3 \to \mathbf{S}^2$ (see Definition 2. 4), and for large enough $i$ we see that the Hopf invariant of $u_i = H(u_i) = 0$. Therefore $u_i$: $\partial C_{r,p} \to \mathbf{S}^2$ is 0-homotopic for large $i$. Since $u_i \to u$ uniformly on $\partial C_{r,p}$, $u$: $\partial C_{r,p} \to \mathbf{S}^2$ is also 0-homotopic and we complete the proof.

Once Proposition 2.5 is proved, the following proposition follows from the same argument in [4].

PROPOSITION 2.6. *Let $u \in W^{1,p}(\boldsymbol{B}^4; \mathbf{S}^2)$ ($\frac{16}{5} \le p < 4$) satisfy $(u^*\omega) \wedge (u^*\omega) = 0$ in $\mathcal{D}'(\wedge^4T^*\boldsymbol{B}^4)$. Then $u$ can be strongly approximated by maps in $C_\varphi^\infty(\boldsymbol{B}^4; \mathbf{S}^2)$.*

By Proposition 2.3 and Proposition 2.6, we easily see that the conclusion of the Theorem holds.

We finally remark that our result also holds in the case where the base manifold is a general four-dimensional Riemannian manifold.

THEOREM 2.7. *Let $M$ be a compact four-dimensional Riemannian manifold without boundary. Then $u \in \tilde{W}^{1,p}(M; \mathbf{S}^2)$ ($\frac{16}{5} \le p < 4$) if and only if $u$ satisfies*

$$(u^*\omega) \wedge (u^*\omega) = 0 \qquad \text{in } \mathcal{D}'(\wedge^4T^*M).$$

In the case where $\partial M \neq \emptyset$, we obtain

THEOREM 2.8. *Let $M$ be a compact four-dimensional Riemannian manifold with non-empty boundary. Let $\varphi: M \to \mathbf{S}^2$ be a given smooth map. Then $u \in W^{1,p}_{\varphi}(M; \mathbf{S}^2)(\frac{16}{5} \leq p < 4)$ can be strongly approximated by maps in $C^\infty_\varphi(M; \mathbf{S}^2)$ if and only if $u$ satisfies*

$$(u^*\omega) \wedge (u^*\omega) = 0 \qquad \text{in } \mathcal{D}'(\wedge^4 T^*M).$$

The idea of the proof of these results is the same as the proof of our main theorem, but is technically more involved. We omit the details.

## APPENDIX

In this section, we prove Theorem A.1, Corollary A.2, and Corollary A.3 which were used in the previous section. These form an $L^p$-version of the classical de Rham–Hodge–Kodaira decomposition theorem.

THEOREM A.1 (see [7]). *Let $1 < p < \infty$ be a real number. Let $\omega \in L^p(\wedge^k T^*\mathbf{R}^n)$ be such that $d\omega = 0$. Then there exists a unique $\alpha \in W^{1,p}(\wedge^{k-1}T^*\mathbf{R}^n)$ satisfying $\omega = d\alpha$ and $d^*\alpha = 0$. Moreover, we have $\|\alpha\|_{W^{1,p}(\mathbf{R}^n)} \leq C_p\|\omega\|_{L^p(\mathbf{R}^n)}$ for some $C_p > 0$ depending only on $p$ and $n$.*

*Proof.* Let $G$ be the fundamental solution of the Laplacian in $\mathbf{R}^n$. Put $\sigma := G * \omega$. Then $\sigma \in W^{2,p}(\wedge^k T^*\mathbf{R}^n)$ and

$$\Delta\sigma = d(d^*\sigma) + d^*(d\sigma) = \omega.$$

Put $\alpha := d^*\sigma$ and $\beta := d\sigma$. Since $d\omega = 0$, we have $\Delta\beta = 0$ and $\beta = 0$. It is easy to see that $\alpha$ satisfies the desired conditions. ∎

COROLLARY A.2. *Let $\Omega$ be a smooth bounded domain in $\mathbf{R}^n$. Let $\omega \in L^p(\wedge^k T^*\mathbf{R}^n)$ be such that $d\omega = 0$ and $\omega|_{\partial\Omega} = 0$. Then there exists $\alpha \in W^{1,p}(\wedge^{k-1}T^*\Omega)$ such that $\omega = d\alpha$, $d^*\alpha = 0$, and $\|\alpha\|_{W^{1,p}(\Omega)} \leq C_p\|\omega\|_{L^p(\Omega)}$. Here $C_p > 0$ is as in Theorem* A.1.

*Proof.* Extend $\omega$ by 0 to $\mathbf{R}^n$ and apply Theorem A.1. ∎

COROLLARY A.3. *Let $\omega \in L^p(\wedge^2 T^*\mathbf{S}^3)$ satisfy $d\omega = 0$. Then there exists a unique $\alpha \in W^{1,p}(\wedge^1 T^*\mathbf{S}^3)$ satisfying $\omega = d\alpha$ and $d^*\alpha = 0$.*

*Moreover, $\|\alpha\|_{W^{1,p}(\mathbf{S}^3)} \leq C_p\|\omega\|_{L^p(\mathbf{S}^3)}$. Here $C_p > 0$ is a constant depending only on $p$.*

*Proof.* We first show the solvability of the equation $\Delta u = \omega$ in $W^{2,p}(\wedge^2 T^*\mathbf{S}^3)$. For this purpose, we need the following *a priori* estimate.

A PRIORI ESTIMATE. *Let $\omega \in L^p(\wedge^2 T^*\mathbf{S}^3)$. Let $u \in W^{2,p}(\wedge^2 T^*\mathbf{S}^3)$ be a solution for the equation $\Delta u = \omega$. Then there exists a constant $C_p > 0$ depending only on $p$ such that the following holds*:

$$\|u\|_{W^{2,p}(\mathbf{S}^3)} \le C_p \|\omega\|_{L^p(\mathbf{S}^3)}.$$

*Proof of a Priori Estimate.* We choose the open cover $\{U_1, U_2\}$ of $\mathbf{S}^3$ such that $U_i \cong \mathbf{R}^3$.

Let $\{\phi_1, \phi_2\}$ be a partition of unity subordinate to $\{U_1, U_2\}$. We may identify $\phi_i u$ with a vector valued function defined in $\mathbf{R}^3$ with compact support. So by the $L^p$-elliptic estimate, we obtain

$$\begin{aligned}\|\nabla^2(\phi_i u)\|_{L^p(U_i)} &\le C_1\|\Delta(\phi_i u)\|_{L^p(U_i)} \\ &\le C_2\{\|\omega\|_{L^p(U_i)} + \|\nabla u\|_{L_p(U_i)} + \|u\|_{L^p(U_i)}\}\end{aligned}$$

for $i = 1, 2$.

By using the interpolation inequality (we may assume $U_i$ is a smooth domain), we obtain

$$\|\varphi_i u\|_{W^{2,p}(U_i)} \le C_3\|\omega\|_{L^p(U_i)} + C_4(\varepsilon)\|u\|_{L^p(U_i)} + \varepsilon\|u\|_{W^{2,p}(U_i)}$$

for any $\varepsilon > 0$.

Therefore we have

$$\|u\|_{W^{2,p}(\mathbf{S}^3)} \le C_p\{\|\omega\|_{L^p(\mathbf{S}^3)} + \|u\|_{L^p(\mathbf{S}^3)}\}. \tag{A.1}$$

Next we claim the following.

CLAIM. *Let $u \in W^{2,p}(\wedge^2 T^*\mathbf{S}^3)$ be a solution for the equation $\Delta u = \omega$, where $\omega \in L^p(\wedge^2 T^*\mathbf{S}^3)$. Then there exists a constant $C_p > 0$ depending only on $p$ such that*

$$\|u\|_{L^p(\mathbf{S}^3)} \le C_p\|\omega\|_{L^p(\mathbf{S}^3)}.$$

*Proof of the Claim.* If the claim is false, there exist sequences $\{u_n\} \subset W^{2,p}(\wedge^2 T^*\mathbf{S}^3)$ and $\{\omega_n\} \subset L^p(\wedge^2 T^*\mathbf{S}^3)$ such that $\|u_n\|_{L^p(\mathbf{S}^3)} = 1$ and $\|\omega_n\|_{L^p(\mathbf{S}^3)} \to 0$ and $\Delta u_n = \omega_n$. By (A.1), we see $\{u_n\}$ is bounded in $W^{2,p}(\wedge^2 T^*\mathbf{S}^3)$. We may assume $u_n \to u$ weakly in $W^{2,p}$ and $u_n \to u$ strongly in $L^p$. Then $\Delta u = 0$ and $\|u\|_{L^p(\mathbf{S}^3)} = 1$.

On the other hand, the space $\mathbf{H}^2$ of harmonic two forms on $\mathbf{S}^3$ is isomorphic to $H^2_{\mathrm{DR}}(\mathbf{S}^3) = 0$ (see [9]), so we have $u = 0$. But this contradicts $\|u\|_{L^p(\mathbf{S}^3)} = 1$. Thus the claim holds. ∎

Combining (A.1) with the above claim, we obtain an *a priori* estimate.

We return to the proof of the solvability. Let $\{\omega_n\} \subset C^\infty(\wedge^2 T^*\mathbf{S}^3)$ be an approximating sequence of $\omega$ in $L^p$. Since $H^2_{\mathrm{DR}}(\mathbf{S}^3) = 0$, by the classical de Rham–Hodge–Kodaira theory, there exist $u_n \in C^\infty(\wedge^2 T^*\mathbf{S}^3)$ such that $\Delta u_n = \omega_n$. By an *a priori* estimate, we see that $\{u_n\}$ is bounded in $W^{2,p}$. Thus there exists $u \in W^{2,p}(\wedge^2 T^*\mathbf{S}^3)$ such that $u_n \to u$ weakly in $W^{2,p}$. Therefore we have $\Delta u = \omega$ and $u$ is a solution. This $u$ satisfies the estimate $\|u\|_{W^{2,p}(\mathbf{S}^3)} \le C_p \|\omega\|_{L^p(\mathbf{S}^3)}$, where $C_p > 0$ is a constant depending only on $p$ and $u$ is a unique solution.

The remaining argument for the proof of Corollary A.3 is essentially the same as in the proof of Theorem A.1 and we omit it here.

*Note.* After I submitted the first version of my paper, Professor F. H. Lin informed me that his student Y. Zhou independently obtained essentially the same result in his unpublished Ph. D. thesis [10]. He treats the case $3 \le p < 4$. But his result is not given in the distributional form. Of course, our result coincides with that of Zhou's when $\frac{16}{5} \le p < 4$.

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